PLASMA STABILITY IN COMBINED MAGNETIC FIELDS

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Recently we have noted increasing interest in toroidal-type systems. One usually attempts to stabilize exchange instabilities in such systems by using a magnetic field configuration with a minimum of $\int H^{-1}\,dl$ at the center. Field configurations of this type naturally lead to a problem in which the magnetic field curvature 1/R has the form

$$1/R = (1 + \varepsilon\cos bz)/\langle R^{-1}\rangle,$$

(the z-axis is along the magnetic field), i.e., to systems with so called "corrugation" of the magnetic-field force lines.

In discussion of the advantages of various configurations, the following question naturally arises: what forms of instability are characteristic for a given configuration of magnetic-field force lines and how unsafe are they?

As in [1,3], the present paper discusses a series of instabilities characteristic of "corrugated" systems. The effect which crossing of the magnetic-field lines of force exerts on an instability is taken into account.

In connection with the question of stabilizing the properties of systems with nonparallel lines of force, it is particularly important to investigate plasma stability for oscillation frequencies $\omega < k_z v_{ti}$ (k_z is the wave vector along the z-axis, v_{ti} is the thermal velocity of the ions). In this range of frequencies only one instability has been found till now: $\partial \ln T_0/\partial \ln n_0 > 2$ ($T_0(x)$ is the temperature, and $n_0(x)$ is the plasma density which is nonuniform in the x direction) [4].

This, together with the proposed (see, for example, [5]) stabilizing effect of Landau damping on the ions, leads one to hope that the plasma will be stable for short-wavelength longitudinal waves. However, it will be shown here that for frequencies $\omega < k_z v_{ti}$ a fairly universal instability exists, and one which develops for short-wavelength waves.

§1. Let us consider the effect on plasma instability of magnetic-field periodicity in the direction of the field (so-called "corrugated systems") and of the crossing of magnetic-field force lines for the case in which perturbations may be assumed to be potential (rot E = 0).

Taking a perturbation of the form $f(x, z)\exp(i\omega t + ik_y y)$, in the magnetohydrodynamic approximation for the ions, we obtain the equation

$$\frac{\partial^2 f}{\partial x^2} - k_y^2 f - \frac{gk_0}{\omega(\omega-\omega_i)} - i\frac{\omega_s}{\omega}\left(1 - \frac{\omega_e}{\omega-\omega_i}\right)\frac{\partial^2 f}{\partial z^2} = 0,$$

$$\omega_s = \frac{\Omega_e\Omega_i}{\nu_e}, \qquad k_0 = \frac{1}{n_0}\frac{\partial n_0}{\partial x}, \qquad g = \frac{v_{ti}^2}{R}. \qquad (1.1)$$

Here Ω_e and Ω_i are the Larmor frequencies of the electrons and ions, respectively, ν_e is the electron collision frequency, ω_e and ω_i are the electron and ion drift frequencies, n_0 is the unperturbed plasma density, and v_{ti} is the thermal velocity of the ions.

The acceleration of the effective force of gravity directed outward from the plasma will be taken to vary periodically along the magnetic field in what follows, so that

$$\frac{1}{R} = \frac{1}{\langle R\rangle}(1 + \varepsilon\cos bz) \qquad \left(\frac{1}{\langle R\rangle} = \int_0^{2\pi}\frac{1}{R}\,d(bz)\right),$$

where R^{-1} and $\langle R^{-1}\rangle$ are the local and average curvatures of the magnetic-field force lines.

We assume that the coefficients in (1.1) are slowly varying functions of x (the transverse coordinate) and we look for a solution of this equation in the form of $f = X(x)Z(z)$. We obtain the following system with the separation constant $\varkappa^2$:

$$X'' + \left(-k_y^2 - \varkappa^2 + k_y^2\frac{gk_0}{\omega\omega^{(1)}} + i\frac{\omega_s\omega^{(2)}}{\omega\omega^{(1)}}k_y^2\Sigma^2x^2\right)X = 0, \qquad (1.2)$$

$$Z'' + \left(i\frac{\varkappa^2\omega\omega^{(1)}}{\omega_s\omega^{(2)}} - i\frac{\varepsilon k_y^2 gk_0}{\omega_s\omega^{(2)}}\cos bz\right)Z = 0. \qquad (1.3)$$

In (1.2) and (1.3) we allow for the effect of the crossing of lines of force arising from the condition

$$k_\parallel = k_z\mathbf{e}_z + \Sigma x k_y\mathbf{e}_y, \qquad \Sigma = d\theta/dx.$$

Here $k_\parallel$ is the wave vector in the direction of the magnetic field, and $\mathbf{e}_z$ and $\mathbf{e}_y$ are unit vectors along the respective axes.

We note that by applying a variational approach to Eq. (1.1), we can reduce the problem immediately to a single equation in the z direction, choosing the perturbation in the x direction correctly. Actually (1.1) can be obtained by variation of the quadratic form

$$M = \frac{1}{2}\int\left[(\nabla_\perp\Phi)^2 - i\frac{\omega_s\omega^{(2)}}{\omega\omega^{(1)}}(\nabla_\parallel\Phi)^2 - \frac{gk_0}{\omega\omega^{(1)}}(\nabla_y\Phi)^2\right]d\tau. \qquad (1.4)$$

Introducing a localized perturbation in the x direction having the form $\sim\exp(-x^2/2\delta^2)$, we obtain

$$M = \int_{-\infty}^{\infty}dx\int\left\{\frac{x^2}{\delta^4}\Phi^2 - k_y^2\Phi^2 - i\frac{\omega_s\omega^{(2)}}{\omega\omega^{(1)}}\left[\Sigma^2x^2\Phi^2k_y^2 + 2\Sigma x\Phi\frac{\partial\Phi}{\partial z}k_y + \right.\right.$$

$$+\left(\frac{\partial\Phi}{\partial z}\right)^2\Big]+\frac{gk_0}{\omega\omega^{(1)}}(k_y{}^2\Phi)^2\Big\}ds,$$

$$\omega^{(1)}\equiv\omega-\omega_i,\qquad \omega^{(2)}\equiv\omega-\omega_i-\omega_e\,. \qquad (1.5)$$

Varying (1.5) with respect to δ and Φ, we obtain the expression

$$\delta^2=\left(i\frac{\omega\omega^{(1)}}{\omega_s\omega^{(2)}\Sigma^2k_y{}^2}\right)^{1/2}, \qquad (1.6)$$

$$\frac{\partial^2\Phi}{\partial z^2}-\Big[-i\frac{\omega\omega^{(1)}}{\omega_s\omega^{(2)}}\left(\frac{1}{2\delta^2}+k_y{}^2\right)+$$

$$+k_y{}^2\frac{\Sigma^2\delta^2}{2}+i\frac{gk_0}{\omega_s\omega^{(2)}}k_y{}^2\Big]\Phi=0 \qquad (1.7)$$

Remembering that $R^{-1} = \langle R^{-1}\rangle(1 + \varepsilon\cos 2z)$, it is clear that relations (1.6) and (1.7) are equivalent to (1.2) and (1.3).

The region δ is essentially associated with the presence of $\Sigma = d\theta/dx$.

The following fact is of interest. We see from (1.5) that for $\omega_i = \omega_e = 0$ this region corresponds to the dimension of the localization region found in [6] by analyzing a fourth-order equation for nonpotential perturbations, which may be dispensed with in the present case. The reason for this is as follows.

When dealing with the equation $\alpha\varphi^{IV} - u_2\varphi'' + u_1\varphi = 0$, it is known (see, for example, [7]) that the second pair of solutions exerts weak influence on the spectrum of eigenvalues for asymptotic solutions $\sim\exp\left(\pm\int(u_2/\alpha)^{1/2}\right)$.

New solutions considered in [6] correspond the solutions given above for potential perturbations (at least for the so-called "gravitational" mode).

We now investigate Eqs. (1.2) and (1.3). The first of these is a Schrödinger-type equation, the second a Mathieu equation.

The eigenvalues of Eq. (1.2) are determined by the relation

$$-k_y{}^2-\varkappa^2+\frac{k_y{}^2gk_0}{\omega\omega^{(1)}}=(2n+1)\left(-i\frac{\omega_s\omega^{(2)}}{\omega\omega^{(1)}}\right)^{1/2}k_y\Sigma$$

$$(n=0,1,2,\ldots)\,. \qquad (1.8)$$

We write Eq. (1.3) in the form

$$Z''+(a-2q\cos 2z)Z=0\,,$$

$$a=i\frac{4}{b^2}\frac{\varkappa^2\omega\omega^{(1)}}{\omega_s\omega^{(2)}},\qquad 2q=i\frac{4\varepsilon k_y{}^2gk_0}{b^2\omega_s\omega^{(2)}}. \qquad (1.9)$$

The condition that the solution of (1.3) should be periodic for $q \leqslant 1$ can then be represented in the form [8]

$$i\frac{\varkappa^2\omega\omega^{(1)}}{\omega_s\omega^{(2)}}=\frac{b^2m^2}{2}-\frac{1}{2(m^2-1)b^2}\left(i\frac{\varepsilon k_y{}^2gk_0}{\omega_s\omega^{(2)}}\right)^2. \qquad (1.10)$$

Relations (1.8) and (1.10) define the dispersion equation for (1.2) and (1.3) in terms of $\varkappa^2$ (m = = 0,1,2,...).

It is clear from (1.8) that if the principal term on the left-hand side is k_y^2, there are no finite solutions. It is assumed here that the term associated with Σ makes the principal contribution to the varying part of the "potential" in Eq. (1.2).

This equation describes the oscillation modes which develop in drift waves in the presence of dissipative factors (the so-called "drift-dissipative" mode) and a gravitational potential (gravitational dissipative mode). The above remark explains why the "drift-dissipative" mode is nonfinite.

For the zeroth harmonic of solution (1.10) (m = 0) and q > 1, the dispersion equation

$$1+\frac{gk_0}{\omega\omega^{(1)}}+\left(i\frac{\omega_s\omega^{(2)}}{\omega\omega^{(1)}}\right)^{1/2}k_y\Sigma=0 \qquad (1.11)$$

has no finite solutions. Here the average curvature has been taken to have the sign of the stable configuration.

Finite solutions exist only for $|gk_0/\omega\omega^{(1)}| > 1$, which corresponds to q > 1. In this case (q > 1), accurate to terms of the order $\sim q^{1/2}$, the dispersion equation has the form

$$-i\frac{\omega\omega^{(1)}}{\omega_s\omega^{(2)}}-i\frac{\Sigma}{k_y}\left(-i\frac{\omega\omega^{(1)}}{\omega_s\omega^{(2)}}\right)^{1/2}+i\frac{gk_0(\varepsilon-1)}{\omega_s\omega^{(2)}}=0\,, \qquad (1.12)$$

or

$$1+\frac{\Sigma}{k_y}\left(-i\frac{\omega_s\omega^{(2)}}{\omega\omega^{(1)}}\right)^{1/2}-\frac{gk_0(\varepsilon-1)}{\omega\omega^{(1)}}=0\,.$$

If 1 is negligibly small in comparison with $gk_0(\varepsilon - 1)/\omega\omega^{(1)}$, provided that $\omega_i \approx \omega_e \approx 0$, we obtain

$$\omega\approx-i\left[\frac{(gk_0)^2(\varepsilon-1)^2}{\omega_s}\frac{k_y{}^2}{\Sigma^2}\right]^{1/3},$$

which corresponds to an aperiodic instability with the given increment and coincides with the results obtained in [1].

The increment of this instability corresponds to the gravitational dissipative mode, but is considerably less than in the absence of longitudinal corrugation.

§2. The question of the stabilizing properties of corrugated systems is of particular importance owing to the fact that systems with crossed magnetic field lines are evidently not very effective. The part played by this factor really comes down to the fact that perturbations with small k_z are very much localized in the transverse direction, and this reduces anomalous diffusion (see [9]). However, the question of plasma stability for perturbations $k_z > \omega/v_{ti}$ still remains.

We will show that such perturbations are unstable for a collisionless regime. The dispersion for a high temperature plasma obtained in [4] is written in the form

$$2+\frac{i\sqrt{\pi}}{k_zv_{te}}(\omega-\omega_e)W\left(\frac{\omega}{k_zv_{te}}\right)\Gamma_0(\eta_e)+$$

$$+\frac{i\sqrt{\pi}}{k_zv_{ti}}W\left(\frac{\omega}{k_zv_{ti}}\right)\Gamma_0(\eta_i)(\omega-\omega_i)=0\,,$$

$$\Gamma_0(\eta)=e^{-\eta}I_0(\eta),\ \eta_e=k_yr_e,\ \eta_i=k_yr_i\,. \qquad (2.1)$$

Here $I_0(\eta)$ is a Bessel function of imaginary argument, and W(x) is a Kramp function. We take asymptotic forms of the functions

$$\Gamma_0(\eta_i) \text{ for } k_y r_i \gtrsim 1, \qquad \Gamma_0(\eta_e) \text{ for } k_y r_e \ll 1,$$
$$W\left(\frac{\omega}{k_z v_{ti}}\right) \text{ for } \omega \ll k_z v_{ti}.$$

As a result we have the following equation:

$$2 + \frac{i\sqrt{\pi}}{k_z v_{te}}(\omega - \omega_e)\left(1 + 2i\frac{\omega}{\sqrt{\pi}\,k_z v_{te}}\right) + \frac{\omega - \omega_i}{\sqrt{2}\,k_y r_i k_z v_{ti}} = 0$$

from which we find that $\mathrm{Im}\,\omega \sim \mathrm{Re}\,\omega \sim k_z \nu_{ti}{}^2 k_y r_i/\omega_i$. From the assumption that $\omega/k_z v_{ti} < 1$, the condition for the existence of a solution is $k_z \ll k_0$.

Let us consider the diffusion which develops for the instability in question. Using the formula for the diffusion coefficient $D \sim \mathrm{Im}\omega\lambda_x^2$ and remembering that in the present case $\lambda_x \sim r_e$ (r_e is the electron Larmor radius), we have $D \sim r_e^2 v_{ti}/r$ (r is the transverse dimension of the system).

Clearly this coefficient is much less than the Baumé diffusion coefficient, but in a high-temperature regime may lead to dangerous particle losses.

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